LECTURE 2: WEAK* LIMITS AND EGOROV'S THEOREM

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Lecture 2 is an introduction to weak* limits of sequences $\mathcal{S} = \{\varphi_{j_k}\}$ of eigenfunctions, or more precisely of their microlocal lifts to T^*M (Wigner distributions). An orthonormal basis tends to zero weakly in L^2, $\varphi_j \rightharpoonup 0$, but of course does not tend to zero strongly. Weak* limits [1] measure the failure of the sequence to be compact in L^2 in a microlocal sense, detecting the phase space points which cause the lack of microlocal compactness. Failure of compactness generally has two sources: (i) concentration, (ii) oscillation. From a quntum mechanics point of view, weak* limits is the study of diagonal matrix elements (expectation values) $\langle A\varphi_j, \varphi_j\rangle$ of observables $A \in \Psi^0(M)$, the algebra of zeroth order pseudo-differential operators. Matrix elements define positive linear functionals

$$\rho_j(A) = \langle A\varphi_j, \varphi_j\rangle$$

on $\Psi^0(M)$. In the high frequency (or semi-classical) limit $\lambda_j \to \infty$, subsequences ρ_{j_k} tend to limit measures on S^*M (the unit co-sphere bundle) which are invariant under the geodesic flow. This is a consequence of Egorov's theorem, proved in Section 3. The weak* limit problem is to determine the invariant measure which arise. Much is known when the geodesic flow is ergodic or completely integrable, and almost nothing is known in other cases. The weak * limit problem is evidently a global problem relating eigenfunctions to the geodesic flow.

We begin by reviewing classical results of Komogorov-Riesz-Tamarkin on compact sequences in $L^2(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is a compact domain. A fundamental difference between general non-compact sequences and the sequence of eigenfunctions φ_{j_k} is that the latter have a fixed frequencies λ_{j_k}, i.e. oscillations can occur only on the wavelength scale $\frac{1}{\lambda_{j_k}}$. General sequences may have oscillations at different scales. Hence, general criteria of compactness are not as useful for eigenfunctions as criteria which take into account the wavelength scale of oscillations and concentrations. These more specific measures of non-compactness are the quantum limits.

On a compact domain in $\mathbb{R}^n$, archetype examples of non-compact sequences are

1. non-compactness due to concentration: $u_n(x) = \epsilon_n^{-d/2}\varphi(\frac{x-x_0}{\epsilon_n})$ where $\varphi \in C_c^\infty(\mathbb{R}^d)$ and $\epsilon_n \to 0$;

2. non-compactness due to oscillation: $u_n(x) = a(x)e^{\frac{2\pi i x\cdot\xi}{h_n}}$. where $h_n \to 0$. More generally, one may consider WKB sequences $u_h(x) = a(x)e^{\frac{i}{h}S(x)}$. Such non-compacness is concentration in Fourier space.

Sequences of eigenfunctions may exhibit both types of non-compactness. Hence, one needs measures of concentration in phase space.

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[1] also known as semi-classical limits, semi-classical defect measures, quantum limits, microlocal defect measures

1. Background on compactness of sequences

We first review the classical results on compact sequences in $L^2(\Omega)$ where Ω is a bounded domain in $\mathbb{R}^n$. The same criteria apply to compact manifolds. In the case of $\mathbb{R}^n$, the compactness criterion may be formulated in terms of the Fourier transforms $\hat{u}_n$. They may then be microlocalized to produce microlocal defects in compactness.

Recall that a subset E of metric space (X, d) is compact if and only if it is complete and totally bounded, that is, for any $r > 0$, E may be covered by a finite number of balls of radius r. Equivalently, E is sequentially compact: every sequence has a convergent subsequence.

1.1. **Kolmogorov compactness criterion.**

Relative compactness of a bounded sequence $\mathcal{S} = \{u_n\}$ in L^2 is a special case of relative compactness of any bounded set in L^2, and means that the L^2 closure is compact. For $L^2(M)$ (M compact) Kolmogorov gave the criterion that

$$\sup_n ||u_n(\cdot + h) - u_n||_{L^2} \to 0, \text{ as h} \to 0 \tag{1}$$

Or for all ϵ there exists δ so that $||u_n \circ T_y - u_n|| < \epsilon$ for all n if $|y| \leq \delta$, where $T_y f(x) = f(x+y)$. That is, small translates of the functions u_n are unifomly close to u_n. On $\mathbb{R}^n$ $\{u_n\}$ is relatively compact on compact sets if and only if

$$\forall \varphi \in C_0^\infty(\Omega), \;\; \sup_n ||(\varphi u_n)(\cdot + h) - \varphi u_n(\cdot)||_{L^2} \to 0, \;\; (h \to 0).$$

Define the L^2 modulus of continuity by

$$\omega_{f,K}(t) = \left(\sup_{|y| \leq t} \int_K |f(x+y) - f(x)|^2 dx \right)^{\frac{1}{2}}.$$

The Riesz-Kolmogorov compactness theorem relates compactness to a unifom L^2 modulus of continuity. Let $K \subset \Omega$ be a compact set which is the closure of an open set. Let $f \in L^2(\Omega)$.

Theorem **1.1.** *Let $K \subset\subset \Omega$. Then $\{u_n\}$ is precompact in $L^2(K)$ if and only if the sequence is uniformly bounded in L^2 and*

$$\omega_{u_n}(t) \leq v(t)$$

for some nondecreasing $v : \mathbb{R}_+ \to \mathbb{R}_+$ with $v(t) \downarrow 0$.

Tamarkin and Riesz generalized Kolmogorov's compactness criterion to sequences on all of $L^2(\mathbb{R}^n)$ by adding a term constraining the mass of the sequence on large balls. We refer to [HH, P85] for the statement. We do not use it because we assume our manifolds or domains are compact.

Proposition **1.2.** *The following is equivalent to* (1)*:*

$$\sup_n \int_{|\xi \geq R} |\hat{u}_n(\xi)|^2 d\xi \to 0, \quad R \to \infty.$$

Proof. This is proved in [P85] and we briefly recap the proof. We work on $\mathbb{R}^n$ and consider a sequence $\{u_n\}$ satisfying (1) or the equivalent versions. Introduce the standard Gaussian $\psi(x) = (2\pi)^{-n/2} e^{-|x|^2/2}$. Let $\psi_R(x) = \psi(Rx)R^n$, so that $\hat{\psi}_R(\xi) = \hat{\psi}(\xi/R)$. Recall that $\hat{\psi}(\xi) = e^{-|\xi|^2/2}, \hat{\psi}_R(0) = 1$.

For $|\xi| \geq 2R, \frac{1}{2} \leq 1 - \hat{\psi}_R(\xi)$, hence for any u_n,

$$\begin{aligned} \tfrac{1}{2}\left(\int_{|\xi|>2R} |\hat{u}_n(\xi)|^2 d\xi\right)^{\frac{1}{2}} &\leq ||\hat{u}_n(\xi)(1-\hat{\psi}_R(\xi)||_2 \\ &\leq C||u_n - u_n * \psi_R||_2 \\ &= C\Big[\int \big|\int \left(\left(u_n(x)-u_n(x-y)\right)\psi_R(y)dy\Big|^2 dx\Big]^{\frac{1}{2}} \\ &\leq C\Bigg[\int \Big[\int \big|(u_n(x)-u_n(x-\tfrac{y}{R})\big|^2 dx]\psi(y)dy\Big|^2 dx\Bigg]^{\frac{1}{2}}. \end{aligned}$$

In the last line we changed variables and applied the Schwarz inequality and Fubini's theorem.

Define the L^2 modulus of continuity of the sequence $\{u_n\}$. It is defined differently from above, in that we sup over the sequence but not over small y.

$$\omega(y) = \sup_n \int |u_n(x) - u_n(x-y)|^2 dx.$$

By assumption, $\omega(y) \to 0$ as $y \to 0$ and $\omega(y) \leq (2M)^2$ for all y. It follows that

$$\tfrac{1}{2}\left(\int_{|\xi|>2R} |\hat{u}_n(\xi)|^2 d\xi\right)^{\frac{1}{2}} \leq 2C\left[\int \omega(\tfrac{y}{R})\psi(y)dy\right]^{\frac{1}{2}} \to 0, \text{ as R} \to \infty.$$

This proves the desired statement. ☐

Since compactness is a 'tail event, an equivalent condition is that

$$\limsup_n \int_{|\xi \geq R} |\hat{u}_n(\xi)|^2 d\xi \to 0, \quad R \to \infty.$$

1.2. **Brief recap on compact operators.** A bounded operator $A \in \mathcal{L}(\mathcal{H})$ on a Hilbert space is *compact* if A takes bounded sequences $\{u_n\}$ to relatively compact sequences. The compact operators form an ideal $\mathcal{K} \subset \mathcal{L}$. Pseudodifferential operators $A \in \Psi{-r}(M)$ of negative order are compact.

Suppose that $\{u_n\}$ is a bounded sequence converging weakly to 0 in $\mathcal{H}$ and that Au_n is relatively compact. We claim that $||Au_n||_{L^2} \to 0$. Indeed, let v be the strong limit of any converging subsequence Au_{n_k}. Then

$$||v||^2 = \langle v, v\rangle = \lim_{k\to\infty} \langle Au_{n_k}, Au_{n_k}\rangle = \lim_{k\to\infty} \langle A^* A u_{n_k}, u_{n_k}\rangle.$$

Now $A^*Au_{n_k} \to A^*v$ strongly, and $||u_{n_k}|| \leq 1$, so

$$\lim_{k\to\infty} \langle A^*Au_{n_k}, u_{n_k}\rangle = \lim_{k\to\infty} \langle A^*v, u_{n_k}\rangle = 0,$$

since $u_{n_k} \to 0$ weakly.

2. MATRIX ELEMENTS

One of the principal techniques for obtaining information on the asymptotics of eigenfunctions is to study matrix elements

$$\rho_{jk}(A) := \langle A\varphi_j, \varphi_k\rangle, \qquad (\rho_j(A) := \langle A\varphi_j, \varphi_j\rangle \text{ when } j = k) \tag{2}$$

of pseudo-differential operators $A \in \Psi^0(M)$ with respect to the eigenfunctions. Since the eigenfunctions are normalized, the linear functionals ρ_{jk} are bounded on the space $\mathcal{L}(\mathcal{H})$ of bounded linear operators on the Hilbert space $\mathcal{H} = L^2(M)$ equipped with the operator norm topology. In quantum mechanics, the functional $\rho_j(A)$ is viewed as the 'expected value of the observable A in the energy state φ_j (of energy λ_j^2).

When an orthonormal basis of eigenfunctions is fixed, we refer to

$$\big(\langle A\varphi_j, \varphi_k\rangle\big)_{j,k=0}^{\infty} \tag{3}$$

as the matrix of A with respect to the orthonormal basis $\{\varphi_j\}$. The matrix of a pseudo-differential operator has special asymptotic properties distinguishing it from the matrix of a general bounded operator.

When we fix a quantization $a \to \mathrm{Op}(a)$ of symbols as pseudo-differential operators, the matrix elements become linear functionals of the symbol and one has a representation,

$$\rho_{j,k}(\mathrm{Op}(a)) = \int_{T^*M} a(x,\xi)\, dW_{j,k} \tag{4}$$

of the linear functional as a distribution $dW_{j,k}$ on smooth symbols. The distribution is sometimes called the Wigner distribution of (φ_j, φ_k) and we follow that terminology here. When $j = k$ we denote $W_{j,k}$ by W_j. In the case of homogeneous pseudo-differential operators, we can view

$$\rho_j(\mathrm{Op}(a)) = \int_{S^*_g M} a\, dW_j$$

as a distribution on the unit cosphere bundle (energy surface). In general, we would like to study the asymptotics of the matrix elements or Wigner distributions for as large as possible a class of symbols or operators. The Wigner distribution is (almost) a positive measure and is truly one if $\mathrm{Op}(a)$ is defined in a certain way.

2.1. Matrix elements and microlocal mass estimates.

If $A = \mathbf{1}_E$ is multiplication by the characteristic function of a nice open set $E \subset M$, then

$$\rho_j(\mathbf{1}_E) = \int_E |\varphi_j|^2 dV_g$$

is the "mass" or the probability that the particle represented by φ_j is located in E. $\mathrm{Op}(\mathbf{1}_E)$ is viewed as the quantization of the characteristic function of a set $E \subset T^*M$. Then $\langle \mathrm{Op}(\mathbf{1}_E)\varphi_j, \varphi_j\rangle$ is the "probability amplitude that the (position, momentum) of the particle is in E." We may regard it as the measure of the microlocal mass of φ_j in E.

Usually we assume that the symbol is C^∞.

2.2. **Diagonal matrix elements as invariant states.** We now develop the view that $\rho_j(A) = \langle A\varphi_j, \varphi_j\rangle$ is an *invariant state* on the C^*-algebra $\Psi^0(M)$.

A 'state' in quantum mechanics might refer to a normalized wave function ψ or to the associated density matrix, namely the linear functional

$$\rho_\psi(A) = \langle A\psi, \psi\rangle$$

on observables $A \in \Psi^0(M)$. This is a standard term in C^* algebras. States have the properties:

- (i) $\rho_\psi(A^*A) \geq 0$;
- (ii) $\rho_\psi(I) = 1$;
- (iii) ρ_ψ is continuous in the norm topology.

The classical analogue of a state on $\Psi^0(M)$ is a probability measure on T^*M. In the case of eigenfunctions, the measures localize on the unit cosphere bundle S^*M, and the probability measures are linear functional on $C(S^*M)$. In particular, eigenfunctions define states, which we denote by

$$\rho_k(A) = \langle A\varphi_k, \varphi_k\rangle. \tag{5}$$

A *state* is a linear functional on $\Psi^0(M)$ such that (i) $\rho_\psi(A^*A) \geq 0$; (ii) $\rho_\psi(I) = 1$; (iii) ρ_ψ is continuous in the norm topology. It is the quantum analogue of a probability measure (a state on $C^0(S^*M)$).

2.3. **Evolution of states: Heisenberg picture.** The evolution of observables in the Heisenberg picture is defined by

$$\alpha_t(A) := U^t A U^{-t}, \quad A \in \Psi^m(M). \tag{6}$$

and since Dirac's Principles of Quantum Mechanics, it was known to correspond to the classical evolution

$$V^t(a) := a \circ g^t \tag{7}$$

of observables $a \in C^\infty(S^*M)$. Egorov's theorem is the rigorous version of this correspondence: it says that α_t defines an order-preserving automorphism of $\Psi^*(M)$, i.e. $\alpha_t(A) \in \Psi^m(M)$ if $A \in \Psi^m(M)$, and that

$$\sigma_{U_t A U_t^*}(x,\xi) = \sigma_A(g^t(x,\xi)) := V^t(\sigma_A), \quad (x,\xi) \in T^*M\backslash 0. \tag{8}$$

It is an immediate consequence of the fact that $U^t\varphi_j = e^{it\lambda_j}\varphi_j$ that the diagonal states ρ_k are invariant under the automorphism α_t (6):

$$\rho_k(U_t A U_t^*) = \rho_k(A). \tag{9}$$

In general, we denote by $\mathcal{E}$ the compact, convex set of states in the vector space of continuous linear functionals on the closure of Ψ^0 in its norm topology. We denote by

$$\mathcal{E}_{\mathbb{R}} = \{\rho \in \mathcal{E} : \rho \circ \alpha_t = \rho\} \tag{10}$$

the compact convex set of invariant states. This is standard notation for invariant states (see e.g. [R]).

Global harmonic analysis exploits the long time behavior of the geodesic flow, e.g. its ergodicity or integrability, to prove results about the high eigenvalue limit of eigenfunctions.

The joint asymptotics $t \to \infty \quad \lambda_j \to \infty$ makes the analysis difficult and the geodesic flow is only a good approximation to the quantum dynamics when

$$|t| \leq T_H(\lambda_j) := \kappa \log \lambda_j,$$

for a certain κ.

2.4. Representations of states as distributions or measures.

if we fix a quantization

$$Op : C^\infty(S^*M) \to \Psi^0(M),$$

the state $\rho_k(A)$ on $\Psi^0(M)$ determines a distribution $d\Phi_j \in \mathcal{D}'(S^*M)$ by the rule,

$$\langle Op(a)\varphi_k, \varphi_k \rangle = \int_{S^*M} a d\Phi_k$$

where $a \in C^\infty(S^*M)$. There exist positive quantizations such as Friedrichs quantization $Op^F(a)$ or anti-Wick quantization [Tay81, Ze87] with the property that if $a \geq 0$ then $Op^F(a) \geq 0$. We denote the associated distribution by $d\Phi_j^F$:

$$\langle Op^F(a)\varphi_k, \varphi_k \rangle = \int_{S^*M} a d\Phi_k^F.$$

Since a positive distribution is a measure, $d\Phi_k^F$ is a probability measure. Such measures form a compact set $\mathcal{M}_1$ in the space $\mathcal{M}$ of positive measures on $C(S^*M)$.

2.5. Weak* limits, quantum limits, semi-classical limit measures.

Proposition **2.1.** *The set $\mathcal{Q}$ of weak* limits of the sequence ρ_k are independent of the choice of quantization, and are invariant probability measures on S^*M for the geodesic flow.*

Proof. The principal symbol σ_A of a pseudo-differential operator is independent of the choice of quantization. Hence, two different quantizations of a differ by a pseudo-differential operator K of negative order. Then K is compact and so $||K\varphi_k||_{L^2} \to 0$.

It also follows that for any $A \in \Psi^0(M)$, any limit of a sequence of $\langle A\varphi_k, \varphi_k \rangle$ is equally a limit of $\langle (A + K)\varphi_k, \varphi_k \rangle$. If ρ_∞ is any weak* limit of the ρ_k then $\rho_\infty(A) = \rho_\infty(A + K)$ and so $\rho_\infty(A) = \int_{T^*M} \sigma_A d\mu$ for some probability measure μ.

The fact that ρ_∞ is boudned on $C^0(S^*M)$ also follows from the fact that for $A \in \Psi^0$, $||\sigma_A||_{L^\infty} = \inf_K ||A + K||$. Hence any weak limit is bounded by a constant times $||\sigma_A||_{L^\infty}$ and is therefore continuous on $C(S^*M)$.

The next property of the ρ_k are consequences of the fact $\rho_k \in \mathcal{E}_{\mathbb{R}}$. Let $\mathcal{M}_I$ be the convex set of invariant probability measures for the geodesic flow G^t. They are also time-reversal invariant. Then $\mathcal{Q} \subset \mathcal{M}_I$ The invariance under g^t follows from $\rho_k \in \mathcal{E}_{\mathbb{R}}$ by Egorov's theorem: any limit of $\rho_k(A)$ is a limit of $\rho_k(Op(\sigma_A \circ g^t))$ and hence the limit measure is G^t invariant.

Regarding the support on S^*M, suppose that $a \in C_0^\infty(T^*M)$ is supported on the set $\{||\xi| - 1| \geq \epsilon\}$. We claim that $||Op(a)\varphi_j||_{L^2} = o(1)$. Indeed, let $b = \frac{a}{|\xi|-1}$. Since a is compactly supported, $a \in S^{-\infty,0}(T^*M)$. Let $B_h = Op_h(b), A_h = Op_h(a)$ with $h = \lambda_j^{-1}$. Then $A_h = B_h(\sqrt{-\Delta} - I) + E_h$ where $E_h \in \Psi^{-\infty,-1}$ and $||E_h||_{L^2 \to L^2} = O(h)$. Hence,

$$||A\varphi_j|| \leq ||B(\lambda_j^{-1}\sqrt{-\Delta} - I)|| + ||E\varphi_j|| = o(1),$$

since $(\lambda_j^{-1}\sqrt{-\Delta} - I)\varphi_j = 0$. □

Since Friedrichs quantization and anti-Wick quantization are somewhat esoteric, we remark that the positivity of the quantum limit distributions also follows easily by Garding's inequality. For the following, we refer to [Zw, Theorem 4.32].

THEOREM **2.2.** *If $a \geq 0$, $a \in S^0(T^*M)$ and $Op(a)$ is any homogeneous quantization of a then*

$$\langle Op(a)\varphi_j, \varphi_j \rangle \geq -C\lambda_j^{-1}.$$

.

Hence, $\lim_{j\to\infty}\langle Op(a)\varphi_j, \varphi_j\rangle \geq 0$.

2.6. **Semi-classical measures and Semiclassical wave front set.** We now give further background on quantum limits, or semi-classical defect measures. A good reference is [Zw, Chapter 5.2]. The supports of these measures are equal to the semi-classical wave front set $WF_h(u_h)$ of a family $\{u_h\}$ of a 'temperted family' [Zw, §8.4.2]. The semi-classical defect measures detect the regions of phase space which cause non-compactness of a sequence of eigenfunctions.

DEFINITION **2.3.** *The semiclassical wave front set $WF_h(u_h)$ of a family u_h is the complement of the (x_0, ξ_0) possessing a neighborhood V and a symbol $a \in S$ with $|a(x_0, \xi_0)| > \delta > 0$ such that*

$$||a^w(x, hD)u_h||_{L^2} = O(h^\infty.$$

In place of L^2 one may use a semiclassical Sobolev norm $H_h(m)$. Equivalently there exist $\varphi, \psi \in C_0^\infty$ with $\varphi = 1$ near x_0, $\psi = 1$ near ξ_0 and

$$||\psi\mathcal{F}_h(\varphi u_h)||_{L^2} = O(h^\infty).$$

If we let $h = h_n$ and $u_h = u_n$ as above, we may ask how $WF_h(u_h)$ is related to $WF(\{u_n\})$.

As the above suggests, one may enlarge the characterization by using general pseudodifferential operators $A \in \Psi^0(M)$ to the sequences. Such A are not compact if their principal symbol σ_A is non-zero. If $\sigma_A \equiv 0$ then $A \in \Psi^{-1}(M)$ and $A \in \mathcal{K}$. But it is possible that σ_A may be supported in a conic subset of $T^*M - 0$ with the property that $A\varphi_{j_k}$ is a pre-compact sequence. This is true if the φ_{j_k} 'micro-localize' in the set

$$\mathrm{Char(A)} := \{(\mathrm{x}, \xi) \in \mathrm{T^*M} : \sigma_{\mathrm{A}}(\mathrm{x}, \xi) = 0\}, \tag{11}$$

so that $A\varphi_{j_k}$ is something like the application of a (micro-locally) compact operator to a weakly converging sequence. For instance,

$$\mathrm{Char(A_R)} = \{(\mathrm{x}, \xi) : \chi(\mathrm{x})\chi_{\mathrm{R}}(\xi) = 0\} \simeq \mathrm{U} \times \mathrm{V}.$$

2.7. **Semi-classical wave front set equals the support of the Microlocal defect measures.**

PROPOSITION **2.4.** *Let $\{u_n\}$ be a bounded sequence in L^2 with a unique microlocal defect measure. Then*

$$WF(u_n)|_{S^*M} = \mathrm{supp}\mu.$$

Proof. We have,

$$||Au_n||^2 = \langle A^*Au_n, u_n\rangle \to \int_{S^*M} |\sigma_A|^2 d\mu.$$

Hence $\mathrm{supp}(\mu) \subset \mathrm{Char(A)}$. If $u_n \to 0$ weakly in L^2 and if Au_n is relatively compact then $||Au_n|| \to 0$. Hence $\mathrm{supp}(\mu) \subset \mathrm{WF(u_n)}$.

Conversely, if $(x, \xi) \notin \mathrm{supp}\mu$, let $a \in S^0$ be such that $a(x, \xi) = 1, |a|^2\mu = 0$. Then $(x, \xi) \notin \mathrm{Char(A)}$ and $||Au_n||_{L^2} \to 0$. Hence Au_n is relatively compact and by the Lemma, $(x, \xi) \notin WF(u_n)$.

□

The converse is also true: If $(x, \xi) \notin \mathrm{supp}\mu$ and if $|a|^2\mu = 0$ then $\langle A^*A(u_n - u), (u_n - u)\rangle \to 0$ and $A(u_n - u)$ is relatively compact in L^2.

Clearly we need to generalize the result to general sequences.

2.8. **Weak * limit problem.** .

One of the best known problems in semi-classical asymptotics is the following:

PROBLEM **2.1.** *Determine the set $\mathcal{Q}$ of 'quantum limits', i.e. weak* limit points of the sequence $\{W_k\}$ of Wigner distributions. Equivalently, determine the set of limit states of $\{\rho_k\}$.*

The following Proposition is all one can say in general:

PROPOSITION **2.5.** *If M is a compact manifold, then $\mathcal{Q} \subset \mathcal{M}_I$, where $\mathcal{M}_I$ is the compact convex set of G^t-invariant probability measures for the geodesic flow. The limits are time-reversal invariant if the eigenfunctions are real valued.*

Any weak * limit of of $\{\rho_k\}$ is an invariant measure for G^t, i.e. $\mu(E) = \mu(G^tE)$. This is because ρ_k is an invariant state for the automorphism:

$$\rho_k(U_tAU_t^*) = \rho_k(A). \tag{12}$$

It follows by Egorov's theorem that any limit of $\rho_k(A)$ is a limit of $\rho_k(\mathrm{Op}(\sigma_A \circ G^t))$ and hence the limit measure is G^t invariant.

There are many invariant probability measures and it is difficult to characterize those which arise as qantum limits. Some examples of invariant measures are:

1. Normalized Liouville measure $d\mu_L$.
2. A periodic orbit measure μ_γ defined by $\mu_\gamma(A) = \frac{1}{L_\gamma}\int_\gamma \sigma_A ds$ where L_γ is the length of γ. A finite sum of periodic orbit measures. In this case the eigenfunctions are sometimes said to 'scar' along γ.ig
3. A delta-function along an invariant Lagrangian manifold $\Lambda \subset S^*M$. The associated eigenfunctions are viewed as *localizing* along Λ.
4. A more general measure which is singular with respect to $d\mu_L$. There are many examples in the hyperbolic case (see e.g. [**?**]).

On a flat torus, for instance,

$$\mathrm{Op}(a)e^{i\langle x, \lambda\rangle} = a(x, \lambda)e^{i\langle x, \lambda\rangle}.$$

Hence

$$\langle \mathrm{Op}(a)e^{i\langle x, \lambda\rangle}, e^{i\langle x, \lambda\rangle}\rangle = \int_{T^n} a\left(x, \frac{\lambda}{|\lambda|}\right) dx.$$

This is Lebesgue measure on the invaraint torus $\xi = \lambda/|\lambda|$. Every Lebesgue measure (i.e., on every invariant torus $\xi = \xi_0$) arises as a weak* limit. For rational tori, eigenvalues are multiple and one may take linear combinations of such exponentials with the same eigenvalue.

2.9. **Matrix elements of spherical harmonics.** We now study matrix elements with respect to spherical harmonics, in particular with respect to the standard basis Y_k^m, i.e. $\langle \mathrm{Op}(a)Y_\ell^m, Y_\ell^m\rangle$ as $\ell \to \infty, m/\ell \to c$.

The image of $T^*S^2 - 0$ under the moment map $\mu(x,\xi) = (p_\theta(x,\xi), |\xi|)$ is a vertical triangular wedge. It is a cone, reflecting that $\mu(x, r\xi) = r\mu(x,\xi)$ is homogeneous. We can break the homogeneity by taking a base for the cone with $|\xi| = 1$, i.e. by considering points $(x, 1)$. This corresponds to looking at $p_\theta : S^*S^2 \to \mathbb{R}$.

Thus, we consider pairs (m_j, ℓ_j) in the joint spectrum of $D_\theta, A = \sqrt{-\Delta + 1/2} - 1/2$ whose projection to the base of the cone has a limit $(c, 1)$.

Theorem **2.6.** *Suppose that* $m_j/\ell_j \to c$. *Then*

$$\langle \mathrm{Op}(a)Y_\ell^m, Y_\ell^m\rangle \to \int_{\mu^{-1}(c,1)} a_0 dx.$$

Thus, the eigenfunctions in this ray localize on the invariant torus $p_\theta^{-1}(c)$.

We define $U(t_1, t_2) = e^{i(t_1 D_\theta + t_2 A)}$ and note that it is a unitary representation of the 2-torus T^2 on $L^2(S^2)$. Further

$$\langle \mathrm{Op}(a)Y_\ell^m, Y_\ell^m\rangle = \langle U(t_1,t_2)^* \,\mathrm{Op}(a) U(t_1,t_2) Y_\ell^m, Y_\ell^m\rangle.$$

Indeed, the eigenvalues cancel out. Average this formula over T^2. We note that

$$\langle A\rangle := \int_{T^2} U(t_1,t_2)^* \,\mathrm{Op}(a) U(t_1,t_2) dt_1 dt_2$$

commutes with both D_θ and A. Indeed, the commutator with A gives $\frac{d}{dt_2}$ under the integral sign, and the integral of this derivative equals zero.

But D_θ, A have a simple joint spectrum: the dimension of the joint eigenspace equals one. Hence, any operator which commutes with them is a function of them. Thus,

$$\langle A\rangle = F(D_\theta, A).$$

The function F must be homogeneous of degree zero. Also, the right side is a ΨDO whose symbol is

$$\langle a_0\rangle : \int_{T^2} a_0(G^{t_1,t_2}(x,\xi))\, dt_1 dt_2.$$

It follows first that

$$\langle \mathrm{Op}(a)Y_\ell^m, Y_\ell^m\rangle = \langle \langle \mathrm{Op}(a)\rangle Y_\ell^m, Y_\ell^m\rangle = F(m,k).$$

Secondly, as $(m_j, \ell_j) \to \infty$ with $m_j/\ell_j \to c$, we have

$$\langle \mathrm{Op}(a)Y_\ell^m, Y_\ell^m\rangle = F(m_j, \ell_j) \to F(c, 1).$$

But also, the limit is the integral of a_0 against an invariant measure. The principal symbol of F is $\langle a_0\rangle$, which is a function on the image of the moment map. Its value at $(c, 1)$ is by definition $\int_{\mu^{-1}(c,1)} a_0 dx$, concluding the proof.

Let us take the 'symbol' of the pair A, L_3. The symbol of A is the metric norm function $|\xi|$ while that of L_3 is the so-called Clairaut integral

$$p_\theta(x,\xi) = \langle \xi, \frac{\partial}{\partial\theta} \rangle.$$

The pair $(p_\theta, |\xi|)$ is called the moment map of the completely integrable geodesic flow of S^2. By the Schwartz inequality, $|p_\theta(x,\xi)| \leq |\frac{\partial}{\partial\theta}| \leq 1$ when $|\xi| = 1$. Hence the image of T^*S^2 under the moment map is a triangular cone in $\mathbb{R}^2$ with vertex at 0 with central axis the y-axis and with sides $y = \pm x$ in the usual $x - y$ coordinates. Compare this to the image of T^*T^2 under the moment map (ξ_1, ξ_2) which is the whole plane.

The quantum limit measures in the case of the standard spheres was determined in [**?**].

Proposition **2.7.** *Every invariant measure for the geodesic flow arises as a weak* limit for a sequence of eigenfunctions on the standard* S^2.

Proof. (Sketch) It suffices to show that every finite sum of delta functions on closed geodesics arises as a quantum limit. Such measures arise by taking linear combinations of the associated Gaussian beams Y^ℓ_ℓ. □

However, one may hope to constrain the possible limits when the geodesic flow is sufficiently chaotic. To do so, one needs to find properties of Wigner measures which are special and which are preserved to some degree in the semi-classical limit. For the remainder of this section, we consider what kinds of properties of eigenfunctions are measured by matrix elements. We also consider matrix elements with respect to more general kinds of operators.

3. Egorov's theorem and invariance of quantum limits

Define the homogeneous symbol class $S^{k,0}(T^*M)$ to consist of smooth functions on $T^*M\backslash 0$ (the puncture cotangent bundle) satisfying the estimates

$$|D^\alpha_x D^\beta_\xi a(x,\xi;h)| \leq C_{\alpha,\beta} \langle \xi \rangle^{k-|\beta|}.$$

Here, $\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$. More generally define $S^{k,m}$ to consist of symbols of the form $h^{-m}a$ where $a \in S^{k,0}$. Symbols in $S^{-\infty,-\infty}$ are called negligeable. The local semi-classical quantization $Op_h(a) = a(x, hD)$ is defined by

$$Op_h(a)(x,y) = (2\pi h)^{-n} \int_{\mathbb{R}^n} a(x,\xi;h) e^{\frac{i}{h}\langle x-y,\xi \rangle} d\xi.$$

the space quantizations of $S^{m,k}$ is denoted $\Psi^{k,n}_h(M)$.

Theorem **3.1.** *Let* A_h *be a semiclassical PsiDO on* M *of order* (k,m) *and let* $U_h(t) = e^{-\frac{it}{h}P_h}$ *where* P_h *is a PsiDO of order* $(1,0)$ *with real principal symbol. Let*

$$A_h(t) = U_h(t)^* A_h U_h(t).$$

Then $A_h(t)$ *is a PsiDO of order* (k,m) *with principal symbol*

$$\sigma_{A_h(t)} = (\Phi^t)^* \sigma_{A_h} := \sigma_{A_h} \circ \Phi^t.$$

There are stronger results:

$$||A_h(t) - Op_h^w(\sigma_{A_h} \circ \Phi_t)||_{L^2 \to L^2} \leq C_1 h e^{\mu t},$$

where C_1, μ are independent of t. The remainder is small if $he^{\mu t}$ is small, or $t \simeq \mu^{-1} \log(\frac{1}{h})$, known as the Ehrenfest time.

Proof. Assume for simplicity that $(k, m) = (0, 0)$. Then,

$$\frac{d}{dt} A_h(t) = \frac{i}{h} U_h(-t)(PA - AP)U_h(t) = \frac{i}{h}[P, A_h(t)].$$

Hence we wish to solve the operator ODE

$$\frac{d}{dt} A_h(t) = \frac{i}{h}[P, A_h(t)], \quad (P = \sqrt{-\Delta}). \tag{13}$$

Assume temporarily that $A_h(t)$ is a PsiDO with principal symbol a_t. Then, taking principal symbols of the equation (13) gives

$$\frac{d}{dt} a_t = \{p, a_t\} = X_p a_t,$$

where X_p is the Hamilton vector field of p. The solution is

$$a_t = a_0 \circ \Phi^t = (\Phi^t)^* a_0,$$

where Φ^t s the Hamilton flow of X_p.

This suggests solving (13) by PsiDO's with well chosen principal symbols. The solution is obtained modulo negligible operators by a recursive procedure which expresses it as an infinite series of improvements plus a remainder. Each term is obtained by solving a principal symbol ODE. The series does not converge but may be summed by a Borel summation procedure. This gives a solution $\widetilde{A}_h(t)$ modulo negligible operators, i.e. smoothing operators of order $O(h^\infty)$. More precisely,

$$\frac{d}{dt} \widetilde{A}_h(t) = \frac{i}{h}[P, \widetilde{A}_h(t)] + O_{L^2 \to L^2}(h^\infty) \tag{14}$$

The true solution $A_h(t)$ is then obtained by variation-of-parameters, showing that $A_h(t) - \widetilde{A}_h(t)$ is negligible Hence, $A_h(t)$ is a PsiDo with the given principal symbol.

The details are as follows. Let $Op(a_t)$ be a quantization of a_t as a PsiDO of order 0. Then,

$$\sigma\left(\frac{d}{dt} Op(a_t) - \frac{i}{h}[P, Op(a_t)]\right) = 0 \implies \frac{d}{dt} Op(a_t) - \frac{i}{h}[P, Op(a_t)] = hR_1(t), \tag{15}$$

where $R_1(t)$ has order -1. Thus, $Op(a_t)$ is an approximate solution of the equation and $Op(a_t)|_{t=0} = A_h + O(h)$.

We now try successive approximations

$$A_n(t) = Op(a_t) + hT_1(t) + h^2 T_2(t) + \cdots + h^n T_n(t) \tag{16}$$

to obtain a better approximate solution. In the second step, we let $A_1(t) = Op(a_t) + hT_1(t)$ and try to solve

$$\frac{d}{dt} A_1(t) = \frac{i}{h}[P, A_1(t)] + h^2 R_2(t). \tag{17}$$

In other words we want the symbol (of order -1) of

$$\frac{d}{dt}A_1(t) - \frac{i}{h}[P, A_1(t)] \tag{18}$$

to be zero. Since $A_1(t) = Op(a_t) + hT_1(t)$ and $\sigma(\frac{d}{dt}A_1(t) - \frac{i}{h}[P, A_1(t)]) = 0$, the symbol of (18) is

$$\begin{array}{l}\sigma_{-1}\left(\frac{d}{dt}Op(a_t) - \frac{i}{h}[P, Op(a(t))]\right) + \sigma_{-1}\left(\frac{d}{dt}T_1(t) - \frac{i}{h}[P, T_1(t)]\right) \\ \\ = \sigma(R_1(t)) + \frac{d}{dt}\sigma_{T_1(t)} - X_p(\sigma_{T_1(t)}.\end{array}$$

This gives an inhomogeneous transport equation for $\sigma_{T_1(t)}$:

$$\frac{d}{dt}\sigma_{T_1(t)} - X_p(\sigma_{T_1(t)}) = \sigma_{R_1(t)}, \quad \sigma_{T_1(t)}|_{t=0} = 0.$$

The solution is

$$\sigma_{T_1(t)} = \int_0^t \sigma_{R_1(s)} \circ \Phi^{t-s} ds. \tag{19}$$

The process may be iterated, each time giving an operator evolution equation of the form

$$\frac{d}{dt}A_j(t) - \frac{i}{h}[P, A_j(t)] = h^{j+1}R_{j+1}(t),$$

and a transport equation for $\sigma_{T_j(t)}$ of the form,

$$\frac{d}{dt}\sigma_{T_j(t)} - X_p(\sigma_{T_j(t)}) = \sigma_{R_j(t)}, \quad \sigma_{T_j(t)}|_{t=0} = 0,$$

with solution (22) with j replacing 1. Each time we are adding a term of lower order in h (or in homogeneity) to $A_h(t)$. The method of asymptotic summation of symbols [Tay81] gives a symbol $\widetilde{a}_t$ and a quantization $\widetilde{A}_h(t) = Op_h(\widetilde{a})$ satisfying

$$\frac{d}{dt}\widetilde{A}_h(t) = \frac{i}{h}[\widetilde{A}_h(t), P] + R_{-\infty}, \quad \widetilde{A}_h(0) - A_h(0) = O(h^\infty).$$

where $R_{-\infty}(t)$ is a semi-classically negligible operator, i.e. of order h^∞ and with a smooth kernel. Here, we use that $\sigma_{T_j}(0) = 0$ for all j, so that the $O(h^\infty)$ term arises only from the asymptotic summation.

Let

$$B_h(t) = A_h(t) - \widetilde{A}_h(t).$$

It satisfies

$$\frac{d}{dt}B_h(t) - \frac{i}{h}[B_h(t), P] + R_{-\infty}, \quad B_h(0) \text{ negligible} \tag{20}$$

By Duhamel's formula (see the Appendix in Section 6), and the fact that $A_h(0) - \widetilde{A}_h(0) = 0$,

$$A_h(t) - \widetilde{A}_h(t) = B_h(t) = \int_0^t U_h^*(t-s)R_{-\infty}(s)U_h(t-s)ds. \tag{21}$$

The second term is of order $O(h^\infty)$ with a smooth kernel by the energy estimates

$$U(t) : H^s(M) \to H^s(M), \quad \forall s, t.$$

Its norm is bounded by $\int_0^t ||R_{-\infty}(s)||ds = O(h^\infty)$.

□

3.1. **More on the homogeneous case.** Here is a slight reworking of the proof using comlete symbols rather than operator equations. The first goal is to construct a complete symbol

$$\widetilde{a}(t,x,\xi) \simeq a_0(t,x,\xi) + a_{-1}(t,x,\xi) + \cdots$$

whose quantization $\widetilde{A}$ in some quantization Op satisfies (14). The composition of symbols is often denoted by $*$ and the complete symbol of the commutator is given by

$$i[P, A(t)] \simeq \sum_{\alpha \geq 0} (D_\xi^\alpha p D_x^\alpha a - D_\xi^\alpha a D_x^\alpha p).$$

We define $a_0(t,x,\xi)$ and $A_0(t) = Op(a_0(t,x,\xi))$ as above. Then,

$$\sigma_{i[P,A_0(t)] - A_0'(t)} = r_0(t,x,\xi) \in S^{-1}.$$

Now construct $a_1(t,x,\xi)$ to solve the inhomogeneous initial value problem,

$$(\frac{\partial}{\partial t} - X_p) a_1(t,x,\xi) = -r_0(t,x,\xi), \;\; a_1(0,x,\xi) = 0.$$

As in (22) the solution is

$$a_1(t,x,\xi) = -\int_0^t r_0 \circ \Phi^{t-s} ds. \tag{22}$$

If $A_1(t) = Op(a_1)$ then[2]

$$\frac{d}{dt}[A_0(t) + A_1(t)] + i[P, A_0 + A_1] \in \Psi^{-2}$$

since the symbol of order zero vanishes and the symbol of order -1 is $r_0 + \dot{a}_1 - X_h a_1 = 0$. So the leading term is of order -2 and so the complete symbol r_2 is of order -2. Then construct a_2 in the same way so that

$$(\frac{\partial}{\partial t} - X_p) a_2(t,x,\xi) = -r_1(t,x,\xi), \;\; a_2(0,x,\xi) = 0.$$

Iterating gives a recursive sequence a_j and an asymptotic sum,

$$\widetilde{a}(t,x,\xi) \sim \sum_{j=0}^\infty a_j(t,x,\xi),$$

satisfying

$$i[P, \tilde{A}(t)] - \tilde{A}'(t) = B_{-\infty} \in \Psi^{-\infty}, \;\; \tilde{A}(0) = A(0).$$

As before, the last step is to show that $B_{-\infty}$ is a smoothing operator, and the proof is exactly as in the previous section.

is smoothing.

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[2] The notation is different from the previous section where the lower A_j's were denoty by T_j

4. H-MEASURES AND MICROLOCAL DEFECT MEASURES

In Section 1 we reviewed some classical results on compactness of uniformly bounded sequences u_h in $L^2(M)$ when M is compact. Usually we assume $u_h \rightharpoonup 0$. Eigenfunctions have a well-defined frequency λ_j, and the semi-classical defect measures and wave front sets are based on setting $h = \lambda_j^{-1}$. There exist more general notions of microlocal defect measures and wave front sets for sequences u_n or u_h where u_n does not have a well-defined frequency. For instance, in the time-dependent setting, u_h may be a more general solution of the wave equation than $e^{it\lambda_j}\varphi_j(x)$ or in the stationary setting it may be a complicated linear combination of eigenfunctions. In this section, we review the definitions and results in this more general setting. It is not really part of the Lecture series on eigenfunctions and will not be referred to again. We follow [Born, Bur, G91, GB].

The basic notions are that of the *sequential wave front set* [3] $WF(\mathcal{S})$ of a bounded sequence of functions in L^2 and that of microlocal defect measures of $\mathcal{S}$. Proposition 2.4 equates $WF(\mathcal{S})$ with the support of the microlocal defect measure μ of $\mathcal{S}$ in the case where $\mathcal{S}$ has a unique microlocal defect measure: $WF(u_n)|_{S^*M} = \mathrm{supp}\mu$. A key point is that if $WF(A)$ is disjoint from $\mathrm{supp}(\mu)$ then $Au_n \to 0$ strongly.

Recall that a set $\mathcal{A}$ in a metric space is totally bounded if it admits a finite ϵ cover for every $\epsilon > 0$. A metric space is compact if and only if it is complete and totally bounded.

4.1. **Microlocal compactness criterion.**

The sequential wave front set is a microlocalization of the Fourier characterization of relatively compact sequences in Proposition 1.2. Instead of using exteriors of balls $|\xi| \geq R$ one uses the intersection with a cone $\Gamma \subset \mathbb{R}^n$.

Definition **4.1.** *Define the sequential wave front set $WF(u_n)$ by the condition: $(x_0, \xi_0) \notin WF(u_n)$ if and only if there exists $\varphi \in C_0^\infty(\Omega)$, $\varphi(x_0) > 0$ and a conic neighborhood Γ of ξ_0 so that*

$$\sup_n \int_{\xi \in \Gamma : |\xi| \geq R} |\widehat{\varphi u_n}(\xi)|^2 d\xi \to 0, \quad R \to \infty. \tag{23}$$

The sequence $\{u_n\}$ is microlocally compact in L^2 at (x_0, ξ_0), or $(x_0, \xi_0) \notin WF(\mathcal{S})$, if there exists a conic neighborhood Γ and a localizing bump function φ so that (23) *holds.*

An equivalent definition is,

$$\limsup_n \int_{\xi \in \Gamma : |\xi| \geq R} |\widehat{\varphi u_n}(\xi)|^2 d\xi \to 0, \quad R \to \infty. \tag{24}$$

By comparison, the semi-classical wave front set is defined by,

$$\limsup_n \int_{\xi \in \Gamma : |\xi| \geq R/h_n} |\widehat{\varphi u_n}(\xi)|^2 d\xi \to 0, \quad R \to \infty. \tag{25}$$

The integral in the definition may be written in terms of matrix elements of pseudodifferential operators. Let $\Gamma = U \times V$ where $U \subset \mathbb{R}^n$ is a small neighborhood around x_0

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[3] This is not a standard term; in [Born] it is called the wave front set of obstructions to microlocal compactness.

and where $V \subset \mathbb{R}^n$ is a cone. Let $A = \chi_R(D)\varphi(x)$, where $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$ and $\chi_R(\xi)$ is the characteristic function of the subset $V_R = \{\xi \in V : |\xi| \geq R\}$. Then $\chi_R(\xi) = \chi_1(\frac{\xi}{R})$. Then

$$\int_{\xi \in \Gamma : |\xi| \geq R} |\widehat{\varphi u}(\xi)|^2 = \langle \chi_R \mathcal{F} \varphi u, \chi_R \mathcal{F} \varphi u \rangle = \langle \varphi \mathcal{F}^* \chi_R(D) \mathcal{F} \varphi u, u \rangle.$$

Here, we should smooth out χ_1 so that it equals 1 in a subcone of V and vanishes outside V, and then $\chi_R^2 \neq \chi_R$ but for expository simplicity we suppress the routine technicalities.

Let $A_R = \varphi \mathcal{F}^* \chi_R(D) \mathcal{F} \varphi$. Then the condition above says that

$$\lim_{R \to \infty} \left(\sup_n \langle A_R u_n, u_n \rangle \right) = 0.$$

Proposition **4.2.**

$$WF(u_n) = \bigcap \mathrm{char(A)},$$

where the intersections runs over all $A \in \Psi^0$ *such that* Au_n *is relatively compact in* $L^2(\Omega)$*.*

Proof. Let $\chi \in S^0(\mathbb{R}^n)$ be supported in Γ and homogeneous for large ξ and satisfies $0 \leq \chi \leq 1$, $\chi(t\xi) = 1$ for large $|t|$. Let $A(x, D) = \chi(D)\varphi(x) \in \Psi^0$. Then $(x_0, \xi_0) \notin \mathrm{Char(A)}$. We have,

$$\begin{array}{rcl} \sup_n \int_{|\xi| \geq R} |\mathcal{F}(Au_n)(\xi)|^2 d\xi & = & \sup_n \int_{|\xi| \geq R} |\chi(\xi)|^2 |\mathcal{F}(\varphi u_n)(\xi)|^2 d\xi \\ & = & \sup_n \int_{\xi \in \Gamma, |\xi| \geq R} |\mathcal{F}(\varphi u_n)(\xi)|^2 d\xi \end{array}.$$

Hence by the Kolmogorov criterion, Au_n is relatively compact. It follows that $(WFu_n)^c \subset (\bigcup \mathrm{Char(A)})^c$.

Conversely, suppose that Au_n is relatively compact and that $(x_0, \xi_0) \notin \mathrm{Char(A)}$. Find a microlocal parametrix B such that $\chi(D)\varphi = B \circ A + R, R \in \Psi^{-\infty}$. Here, $\chi \equiv 1$ in a small conic neighborhood Γ of (x_0, ξ_0). Then $\chi(D)\varphi u_n$ is compact and

$$\begin{array}{rcl} \sup_n \int_{\xi \in \Gamma, |\xi| \geq R} |\mathcal{F}(\varphi u_n)(\xi)|^2 d\xi & \leq & \sup_n \int_{|\xi| \geq R} |\chi(\xi)|^2 |\mathcal{F}(\varphi u_n)(\xi)|^2 d\xi \\ & = & \sup_n \int_{|\xi| \geq R} |\mathcal{F}(\chi(D)\varphi u_n)(\xi)|^2 d\xi \end{array}.$$

Hence $(WF(u_n))^c \subset \bigcup (\mathrm{Char(A)})^c$.

□

5. Garding's inequality

We prove Garding's inequality for positive pseudo-differential operators in $\Psi^0(M)$. The extension to operators of general orders is immediate by composing with powers $(-\Delta)^s$ of the Laplacian. We write $p(x, D) = Op(p)$ for some quantization.

Theorem **5.1.** *Let* M *be compact and let* $A \in \Psi^0(M)$ *and suppose that* $\Re p(x, \xi) \geq C > 0$ *for* $|\xi|$ *sufficiently large. Then for any* $s \in \mathbb{R}$ *and any* $u \in C^\infty(M)$*,*

$$\Re \langle p(x, D)u, u \rangle \geq C ||u||_{L^2}^2 - C' ||u||_{H^s}.$$

Proof. For simplicity assume $p(x,\xi) \in \mathbb{R}$. Let $r(x,\xi) = p(x,\xi) - C$. Using symbol calculus there exists $B \in \Psi^0$ so that $r(x,D) - C = B^*B + S$ where $S \in \Psi^{-\infty}$. Hence,

$$\langle p(x,D)u, u\rangle = C||u||^2_{L^2} + ||Bu||^2 + \langle Su, u\rangle.$$

But $|\langle Su, u\rangle| \leq C'||u||_{H^s}$. □

6. Appendix on Duhamel's formula

Suppose that $S(t)$ is the solution operator for the initial value problem

$$\begin{cases} W_t + AW = 0, \\ U(0) = \Phi \end{cases}$$

Then the solution of the inhomogeneous problem

$$\begin{cases} W_t + WU = F, \\ W(0) = \Phi \end{cases}$$

is given by

$$W(t) = S(t)\Phi + \int_0^t S(t-s)F(s)ds. \tag{26}$$

Now consider the inhomogeneous initial value problem for the wave equation:

$$\begin{cases} u_{tt} - \Delta u = f(x,t), \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x). \end{cases}$$

Proposition 6.1. *In the case of* $\mathbb{R}^3$, *the solution is*

$$\begin{aligned} u(x,t) &= \tfrac{1}{4\pi t^2} \int_{\partial B(x,t)} [\varphi(y) + \nabla\varphi(y)\cdot(y-x) + t\psi(y)]dS(y), \\ &+ \int_0^t \tfrac{1}{4\pi(t-s)} \int_{\partial B(x,t-s)} f(y,s)dS(y)ds. \end{aligned}$$

Proof. Convert this to a first order system:

$$\begin{cases} \begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix}. \\ \begin{pmatrix} u \\ v \end{pmatrix} |_{t=0} = \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \end{cases}$$

Let

$$W = \begin{pmatrix} u \\ v \end{pmatrix}, \quad A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ f \end{pmatrix}, \quad \Phi = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}.$$

In the case of $\mathbb{R}^3$, Kirchhoff's formula gives the solution of the homogeneous IVP as

$$u(x,t) = \frac{1}{4\pi t^2} \int_{\partial B(x,t)} [\varphi(y) + \nabla\varphi(y)\cdot(y-x) + t\psi(y)]dS(y).$$

Hence the solution of the first order system is

$$S(t)\begin{pmatrix}\varphi\\ \\ \psi\end{pmatrix}=\left[\begin{array}{l}\frac{1}{4\pi t^2}\int_{\partial B(x,t)}[\varphi(y)+\nabla\varphi(y)\cdot(y-x)+t\psi(y)]dS(y),\\ \\ \partial_t\left(\frac{1}{4\pi t^2}\int_{\partial B(x,t)}[\varphi(y)+\nabla\varphi(y)\cdot(y-x)+t\psi(y)]dS(y)\right)\end{array}\right].$$

Hence,

$$S(t-s)F(s)=S(t-s)\begin{pmatrix}0\\ \\ f(s)\end{pmatrix}=\left[\begin{array}{l}\int_0^t\frac{1}{4\pi(t-s)}\int_{\partial B(x,t-s)}f(y,s)dS(y)ds,\\ \\ \partial_t\left(\int_0^t\frac{1}{4\pi(t-s)}\int_{\partial B(x,t-s)}f(y,s)dS(y)ds\right)\end{array}\right.$$

Since

$$\begin{pmatrix}u(x,t)\\ \\ v(x,t)\end{pmatrix}=W(x,t)=S(t)\Phi(x)+\int_0^t S(t-s)F(x,s)ds,$$

we get the stated formula.

□

7. Appendix on the action of a pseudo-differential operator on an exponential

In this supplement we state an important Lemma on the action of homogeneous $A\in\Psi^0(M)$ on semi-classical Lagrangian states $u(x,\tau):=a(x)e^{i\tau\varphi(x)}$. Here $\tau=h^{-1}$ is a semi-classical parameter. It is called the Fundamental Asymptotic Expansion Lemma in [Tay81, VIII, Section 7]. It is also Lemma 2.11 of [Ho68].

A simple example is where $\varphi(x)=\langle x,\xi\rangle$. By definition,

$$Ae^{i\langle x,\xi\rangle}=a(x,\xi)e^{i\langle x,\xi\rangle},$$

where $a(x,\xi)$ is the complete symbol of A. If we write $\xi=\tau\omega$ where $\omega=\frac{\xi}{|\xi|}$ then,

$$Ae^{i\tau\langle x,\omega\rangle}=a(x,\tau\omega)e^{i\tau\langle x,\omega\rangle},$$

and if $A\in\Psi^0(M)$ then $a(x,\tau\omega)$ has a polyhomogeneous expansion in τ. The same is true for any phase function φ.

We use the following notation:

$$\varphi(y)=\varphi(x)+\langle y-x,\nabla\varphi(x)\rangle+\rho_x(y).$$

Proposition 7.1. *Suppose that $d\varphi(x)\neq 0$ in the support of $a(x)$. Then,*

$$e^{-i\tau\varphi}p(x,D)(ae^{i\tau\varphi(x)})=\sum_{|\alpha|\le N}\frac{1}{\alpha!}p^{(\alpha)}(x,\tau d\varphi(x))D_y^\alpha(a(y)e^{i\lambda\rho_x(y)}++\tau^{-N/2}R_N(x,\tau),$$

where R_N is uniformly bounded for $\tau\geq 1$.

The notation $p^{(\alpha)}$ means $D_\xi^\alpha p$. The first term is $p_0(x,\tau d\varphi(x))a(x)$ where $p_0=\sigma_P$. The proof amounts to representing $p(x,D)$ as a Fourier integral, integrating against $u(x,\tau)$ and applying stationary phase.

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